

A note on the scale dependence of the Burkardt sum rule

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Abstract

In this short note, we argue that the Burkardt sum rule for the Sivers functions can be used to check the consistency of evolution equations of three-parton correlators.

Sum rules are often very useful for providing model independent constraints for non-perturbative objects. Among them, the Burkardt sum rule [1] is of particular interest in studying transverse single spin asymmetry(SSA) phenomenology. It states that the net transverse momentum carried by partons inside a transversely polarized nucleon vanishes when summing over all parton flavors,

$$\sum_{a=q,\bar{q},g} \langle k_{\perp}^a \rangle = 0 \quad (1)$$

Expressed in terms of the Sivers functions [2] it takes form,

$$\sum_{a=q,\bar{q},g} \int_0^1 dx \int d^2 k_{\perp} k_{\perp}^2 f_{1T}^{\perp a}(x, k_{\perp}^2) = 0 \quad (2)$$

The Burkardt sum rule was first derived in Ref. [1], and previously has been proved in an alternative way by Lorce [3]. It has been checked in various model calculations [4–8].

Using the well known tree level relations between the k_{\perp}^2 moment of the Sivers functions and the corresponding collinear twist-3 correlations [9], the sum rule can be re-expressed as,

$$\int_0^1 dx T_G^{(+)}(x, x) + \sum_{a=q,\bar{q}} \int_0^1 dx T_F^a(x, x) = 0 \quad (3)$$

where T_F is the Qiu-Sterman(QS) function [10, 11], and $T_G^{(+)}$ is the C-even tri-gluon correlation [12–14]. The scale dependence of the sum rule thus can be investigated by using the scale evolution equations of the relevant twist-3 correlations T_F and $T_G^{(+)}$ which already existed in the literatures [15–26]. Note that this subject has also been studied from a different aspect of view in Ref. [27].

We start from the scale evolution equation for quark QS function $T_F^q(x, x)$ [15–23],

$$\begin{aligned} \frac{\partial T_F^q(\xi, \xi, \mu^2)}{\partial \ln \mu^2} \Big|_{q,\bar{q} \rightarrow q} = & \frac{\alpha_s}{2\pi} \int_{\xi} \frac{dx}{x} \left[C_F \left\{ \frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right\} T_F^q(x, x) \right. \\ & + \frac{C_A}{2} \left\{ \frac{1+z}{1-z} T_F^q(\xi, x) - \frac{1+z^2}{1-z} T_F^q(x, x) - 2\delta(1-z) T_F^q(x, x) \right\} \\ & \left. - \frac{N_c}{2} \tilde{T}_F^q(\xi, x) + \frac{1}{2N_c} (1-2z) T_F^q(\xi, \xi-x) - \frac{1}{2N_c} \tilde{T}_F^q(\xi, \xi-x) \right] \end{aligned} \quad (4)$$

where $z = \xi/x$. The last two terms in the above formula contain anti-quark contribution.

We proceed by carrying out the integration over ξ on both sides of the above equation,

$$\begin{aligned} \frac{\partial T_F^q(\mu^2)}{\partial \ln \mu^2} \Big|_{q, \bar{q} \rightarrow q} &= \frac{\alpha_s}{2\pi} \frac{C_A}{2} \left\{ \int dx dz \frac{1+z}{1-z} T_F^q(zx, x) - T_F^q(\mu^2) \int dz \frac{1+z^2}{1-z} \right. \\ &\quad \left. - \int dx dz \tilde{T}_F^q(zx, x) - 2T_F^q(\mu^2) \right\} \\ &\quad + \frac{\alpha_s}{2\pi} \frac{1}{2N_c} \int dx dz \left\{ (1-2z) T_F^q(zx, (z-1)x) - \tilde{T}_F^q(zx, (z-1)x) \right\} \quad (5) \end{aligned}$$

where we introduce the short hand notation $T_F^q(\mu^2) = \int d\xi T_F^q(\xi, \xi, \mu^2)$. The same analysis applies to the scale evolution of anti-quark QS function $T_F^{\bar{q}}(\mu^2)$. Note that our convention for $T_F^{\bar{q}}$ differs from the one used in Ref. [15] by a minus sign, such that the relation between the k_\perp^2 moment of the anti-quark Siverson function and $T_F^{\bar{q}}$ is the same as the relation for the quark ones in a same process (for example, SIDIS process). Combining quark and anti-quark contributions, one obtains,

$$\begin{aligned} \frac{\partial [T_F^q(\mu^2) + T_F^{\bar{q}}(\mu^2)]}{\partial \ln \mu^2} \Big|_{q, \bar{q} \rightarrow q, \bar{q}} &= \frac{\alpha_s}{2\pi} \frac{C_A}{2} \left\{ \int dx dz \frac{1+z}{1-z} [T_F^q(zx, x) + T_F^{\bar{q}}(zx, x)] \right. \\ &\quad \left. - \int dz \frac{1+z^2}{1-z} [T_F^q(\mu^2) + T_F^{\bar{q}}(\mu^2)] \right. \\ &\quad \left. - \int dx dz [\tilde{T}_F^q(zx, x) + \tilde{T}_F^{\bar{q}}(zx, x)] - 2 [T_F^q(\mu^2) + T_F^{\bar{q}}(\mu^2)] \right\} \quad (6) \end{aligned}$$

To arrive the above result, we have made use of the symmetry properties $T_F^{q, \bar{q}}(x_1, x_2) = T_F^{q, \bar{q}}(x_2, x_1)$, $\tilde{T}_F^{q, \bar{q}}(x_1, x_2) = -\tilde{T}_F^{q, \bar{q}}(x_2, x_1)$ and the relations $T_F^q(x_1, -x_2) = T_F^{\bar{q}}(-x_1, x_2)$, $\tilde{T}_F^q(x_1, -x_2) = \tilde{T}_F^{\bar{q}}(-x_1, x_2)$ [15, 18].

The scale evolution of the tri-gluon correlation $T_G^{(+)}$ also receives the contribution from quark and anti-quark [18, 24]

$$\begin{aligned} \frac{\partial T_G^{(+)}(\xi, \xi, \mu^2)}{\partial \ln \mu^2} \Big|_{q, \bar{q} \rightarrow g} &= \frac{\alpha_s}{2\pi} \sum_{q, \bar{q}} \int_\xi \frac{dx}{x} \frac{C_A}{2} \left\{ \frac{1+(1-z)^2}{z} [T_F^q(x, x) + T_F^{\bar{q}}(x, x)] \right. \\ &\quad \left. - \frac{2-z}{z} [T_F^q(x, x-\xi) + T_F^{\bar{q}}(x, x-\xi)] \right. \\ &\quad \left. + [\tilde{T}_F^q(x-\xi, x) + \tilde{T}_F^{\bar{q}}(x-\xi, x)] \right\} \quad (7) \end{aligned}$$

Carrying out the integration over ξ on both sides of the equation and changing the integration

variable $z \rightarrow 1 - y$,

$$\begin{aligned} \frac{\partial T_G^{(+)}(\mu^2)}{\partial \ln \mu^2} \Big|_{q, \bar{q} \rightarrow g} &= \frac{\alpha_s}{2\pi} \sum_{q, \bar{q}} \frac{C_A}{2} \left\{ [T_F^q(\mu^2) + T_F^{\bar{q}}(\mu^2)] \int dy \frac{1+y^2}{1-y} \right. \\ &\quad + \int dx \int dy \left(-\frac{1+y}{1-y} \right) [T_F^q(x, yx) + T_F^{\bar{q}}(x, yx)] \\ &\quad \left. + \int dx \int dy [\tilde{T}_F^q(yx, x) + \tilde{T}_F^{\bar{q}}(yx, x)] \right\} \end{aligned} \quad (8)$$

Adding up Eq. 6 and Eq. 8, we obtain,

$$\frac{\partial \sum_{q, \bar{q}} [T_F^q(\mu^2) + T_F^{\bar{q}}(\mu^2) + T_G^{(+)}(\mu^2)]}{\partial \ln \mu^2} \Big|_{q, \bar{q} \rightarrow q, \bar{q}, g} = -\frac{\alpha_s}{2\pi} \sum_{q, \bar{q}} C_A [T_F^q(\mu^2) + T_F^{\bar{q}}(\mu^2)] \quad (9)$$

We now consider the contribution from the tri-gluon correlation $T_G^{(+)}(x, x)$ to the scale evolution of the quark and anti-quark QS functions. It is worthy to mention that the O type tri-gluon correlation (related to the function $T_G^{(-)}(x, x)$) contributes the same in magnitude and opposite in sign to both the quark and anti-quark Sivers functions. This is not surprising because the O type tri-gluon relation is related to the k_\perp moment of the spin dependent odderon which measures the difference between particle and anti-particle scattering [28]. Therefore, to study the scale dependence of the sum rule, we only need to consider $T_G^{(+)}(x, x)$ contribution which has been given in Refs. [15, 21, 26],

$$\frac{\partial T_F^q(\xi, \xi, \mu^2)}{\partial \ln \mu_F^2} \Big|_{g \rightarrow q} = \frac{\alpha_s}{2\pi} \int_\xi \frac{dx}{x} \frac{1}{2} [z^2 + (1-z)^2] T_G^{(+)}(x, x) \quad (10)$$

which leads to

$$\frac{\partial T_F^q(\mu^2)}{\partial \ln \mu_F^2} \Big|_{g \rightarrow q} = \frac{\alpha_s}{2\pi} \frac{1}{3} T_G^{(+)}(\mu^2) \quad (11)$$

The same equation applies to the anti-quark case.

We move on to discuss the contribution from the tri-gluon correlation $T_G^{(+)}$ to the scale evolution of itself. Three different evolution equations have been derived by three groups, respectively [15, 18, 25]. The possible source of the discrepancy between the results in Refs. [15, 25] is that the different parametrization for the tri-gluon correlations have been used [13, 14]. We are not able to localize the source of the discrepancy between the results [18, 25], as the authors of paper [18] derived the evolution equations in a very different formalism.

Here, we use the one obtained in Ref. [25],

$$\begin{aligned} \frac{\partial \left[\frac{N(\xi, \xi) - N(\xi, 0)}{\xi} \right]}{\partial \ln \mu_F^2} \Big|_{g \rightarrow g} = & \frac{\alpha_s}{2\pi} C_A \int_{\xi}^1 \frac{dx}{x^2} \left\{ \frac{(z^2 - z + 1)^2}{z(1-z)_+} [N(x, x) - N(x, 0)] \right. \\ & + \frac{1+z^2}{2z(1-z)_+} N(\xi, x) - \frac{1+(1-z)^2}{2z(1-z)_+} N(x, x-\xi) \\ & \left. - \frac{z^2 + (1-z)^2}{2z(1-z)_+} N(\xi, \xi-x) - \delta(1-z) [N(x, x) - N(x, 0)] \right\} \\ & + \frac{\alpha_s}{2\pi} \left(C_A \frac{11}{6} - \frac{n_f}{3} \right) [N(\xi, \xi) - N(\xi, 0)] \end{aligned} \quad (12)$$

where the combination of the N type tri-gluon correlation defined in Ref. [14] can be related to $T_G^{(+)}$ through the following identity,

$$\frac{x}{2\pi} T_G^{(+)}(x, x) = -4M_N [N(x, x) - N(x, 0)] \quad (13)$$

Carrying out the integration over ξ on the both sides of Eq. 12, we obtain,

$$\frac{T_G^{(+)}(\mu^2)}{\partial \ln \mu_F^2} \Big|_{g \rightarrow g} = -\frac{\alpha_s}{2\pi} C_A T_G^{(+)}(\mu^2) - \frac{\alpha_s}{2\pi} \frac{n_f}{3} T_G^{(+)}(\mu^2) \quad (14)$$

To arrive the above compact expression, we have used the same mathematic trick, i.e. changing integration variable $z \rightarrow 1-y$, as well as the symmetry properties for the N type tri-gluon correlation: $N(x_1, x_2) = N(x_2, x_1)$, $N(x_1, x_2) = -N(-x_1, -x_2)$ [14].

Collecting the intermediate results presented in Eq. 9, Eq. 11, and Eq. 14, one obtains,

$$\frac{\partial \sum_{q, \bar{q}, g} [T_F^q(\mu^2) + T_F^{\bar{q}}(\mu^2) + T_G^{(+)}(\mu^2)]}{\partial \ln \mu^2} = -\frac{\alpha_s}{2\pi} C_A \sum_{q, \bar{q}, g} [T_F^q(\mu^2) + T_F^{\bar{q}}(\mu^2) + T_G^{(+)}(\mu^2)] \quad (15)$$

It is worthy to mention that all terms on the right side of the equation entirely come from the so-called boundary terms first discovered by Braun, Manashov and Pirnay [18]. We reexpress the above formula as,

$$\frac{\partial \left[\sum_{a=q, \bar{q}, g} \langle k_{\perp}^a \rangle(\mu^2) \right]}{\partial \ln \mu^2} = -\frac{\alpha_s}{2\pi} C_A \sum_{a=q, \bar{q}, g} \langle k_{\perp}^a \rangle(\mu^2) \quad (16)$$

which can be readily solved,

$$\sum_{a=q, \bar{q}, g} \langle k_{\perp}^a \rangle(Q^2) = e^{-\frac{C_A}{2\pi} \int_{Q_0^2}^{Q^2} \alpha_s(\mu^2) \frac{d\mu^2}{\mu^2}} \sum_{a=q, \bar{q}, g} \langle k_{\perp}^a \rangle(Q_0^2) \quad (17)$$

This is the main result of our short note. It is easy to see that the Burkardt sum rule holds at any scale provided that it is satisfied at one arbitrary scale. If the Burkardt sum rule were

violated at a low initial scale for some unknown reason, judging from the above formula, it will be asymptotically satisfied at higher scale.

To summarize: We found that the Burkardt sum rule is stable under QCD scale evolution if one uses the tri-gluon evolution equation obtained in Ref. [25]. However, without making any model dependent assumption for the function $T_G^{(+)}$, we are not able to show that the Burkardt sum rule is preserved under QCD scale evolution when the different tri-gluon evolution equations are used [15, 18]. These findings might shed new light on the controversy about the tri-gluon scale evolution.

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